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ELECTRON CAPTURE IN STELLAR INTERIORS*

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ABSTRACT

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The reasons why nuclear electron-capture rates in stars depend on temperature and density are discussed, and some astrophysical applications of continuum electron-capture rates are reviewed. The modern theory of nuclear beta-decay is then used to calculate stellar continuum electron-capture rates for transitions of an arbitrary degree of forbiddenness. The equations that are most useful for astrophysical applications are discussed in detail; particular emphasis is placed upon methods for predicting stellar rates that utilize, whenever possible, terrestrial measurements. Three examples are discussed that illustrate the use of the formulae given in this paper; the examples are: (a) the electron-capture lifetime of a proton, (b) the stellar beta-decay of K^{40}, and (c) the effect of forbidden transitions on the abundances of elements in the iron peak. AUTHOR

I. INTRODUCTION

Section a) of this introduction is devoted to answering two questions: (1) why do electron-capture lifetimes of nuclei in a star depend sensitively on the local temperature and density, and (2) what are some of the astrophysical applications of stellar electron-capture lifetimes? Section b) of the introduction is a summary of the results presented in the remainder of this paper.

a) Motivation

Atoms in stellar interiors are highly ionized and hence cannot capture bound electrons as easily as they can on earth. This high degree of ionization is largely a consequence of the enormous temperatures that exist in stellar interiors. For example, temperatures in the range 10^{+8} to 10^{+9} ^{o}K are thought to obtain in the interiors of red giant stars (Burbidge, Burbidge, Fowler, and Hoyle 1957; Cameron 1958); this temperature range corresponds to an average thermal energy of 13-130 kev. Since the ionization energy of a 1s electron in Californium (Z = 98) is about 133 kev, it is obvious that most nuclei in the interior of a red giant possess few, if any, bound electrons. Moreover, the thermal energy in the interior of a main sequence star is of the order of 1 kev, so that light nuclei, such as He^3 or Be^7, are completely stripped of electrons in the interiors of main sequence stars. The above qualitative arguments are supported by the quantitative analysis of Cox and Eilers (1962), who calculated, from statistical mechanical considerations, the average degree of ionization of a number of heavy elements under some typical stellar-interior conditions.

Nuclei that decay on earth by the capture of bound atomic electrons may decay in stellar interiors by the capture of free electrons from the surrounding hot plasma. The terrestrial capture of a bound electron is described symbolically by the following equation:

$$e_b + (Z,A) \rightarrow (Z-1,A) + \nu \quad , \tag{1}$$

where Z and A are the nuclear charge and atomic number, respectively, of the initial nucleus. The corresponding stellar reaction is written symbolically

$$e_c + (Z,A) \rightarrow (Z-1,A) + \nu \quad , \tag{2}$$

where e_c is any continuum electron in the plasma surrounding (Z,A). Under most stellar-interior conditions, reaction (2) is faster than reaction (1) (Schatzman 1958; Bahcall 1962 a,b; Fowler and Hoyle 1963).

The rate of reaction (2) is proportional to the probability that a continuum electron is present at the nucleus where it can be captured. The probability of finding an electron at the nucleus is in turn proportional to the electron density and inversely proportional to the average electron velocity, which for nondegenerate electrons depends on the square root of the temperature. Thus the rate of reaction (2) depends strongly on the local electron temperature and density.

If a nucleus decays terrestrially by positron emission,

$$(Z,A) \rightarrow (Z-1,A) + e^{+} + \nu \quad , \tag{3}$$

it will decay primarily by continuum electron capture in a star whose density is sufficiently high. This change in the mode of decay occurs because electron-capture probabilities are enhanced by high densities and most positron emission rates are insensitive to densities of magnitude encountered in stars (Bahcall 1962a).

If a nucleus (Z,A) is a stable beta-decay product on earth,

$$(Z-1,A) \rightarrow (Z,A) + e^{-} + \bar{\nu} \quad , \tag{4}$$

it can undergo induced electron capture at sufficiently high temperatures or densities via reaction (2). The rates of such endoergic reactions, for example,

$$e_c^{-} + p \rightarrow n + \nu \quad ,$$

are extremely sensitive functions of temperature and density since they only occur for continuum electrons having at least the threshold energy.

Terrestrial electron captures take place via reaction (1) and their rates are therefore largely determined by fixed atomic parameters. Stellar electron captures take place primarily via reaction (2) and hence depend on the variable properties of the stellar plasma.

Thus stellar-capture rates are highly variable, although terrestrial-capture rates are almost immutable. This difference in behavior is due ultimately to the enormous range of temperatures and densities that occur in stars.

From the time a star first begins to use nuclear reactions as an energy source until its final stages of evolution, electron capture plays an important role in the nuclear transformations responsible for stellar energy generation and the synthesis of heavy elements. Some applications of electron-capture calculations are reviewed below.

Schatzman (1958) has discussed the role of He^3 electron capture and Be^7 electron capture in the proton-proton chain. As an interesting side light, we note that an accurate calculation of the Be^7 decay rate in the sun (Bahcall 1962b) has recently been combined with other nuclear and astronomical data to obtain a prediction of the solar neutrino flux (Bahcall, Fowler, Iben, and Sears 1963, hereafter referred to as FIBS).

A knowledge of the beta-decay rates (including electron-capture rates) of heavy nuclei under extreme conditions of temperature and density is necessary for a detailed understanding of the formation of heavy elements by slow neutron capture (Clayton, Fowler, Hull, and Zimmerman 1961; Cameron 1959a). If the beta-decay lifetime of an unstable isotope is long compared to its neutron-capture lifetime, the isotope will usually capture a neutron instead of transforming by some beta-decay process. Conversely, if the beta-decay lifetime of an unstable isotope is short compared to its neutron-capture lifetime, the nucleus will usually beta-decay before a neutron is captured. Thus the path of successive neutron captures in the isotope chart, and hence the heavy element abundances, are determined by beta-decay rates as well as neutron capture cross sections.

Salpeter (1961) has recently emphasized that electron capture in low-mass, low-temperature stars can significantly affect their chemical composition, and hence their mean molecular weight. In order to compute the correct chemical compositions for an evolutionary sequence of such low-mass stellar models, it is necessary to have theoretical expressions for the relevant electron-capture lifetimes.

Fowler and Hoyle (1963) have recently performed a massive reinvestigation of the equilibrium process that is assumed to be responsible for the formation of heavy elements in the iron peak. At the high temperatures thought to obtain during the formation of the iron-peak isotopes, a great variety of nuclear reactions occur. These prompt nuclear reactions quickly bring the nuclei into a quasi-equilibrium state in which the most abundant nuclei have an approximately equal number of neutrons and protons and also have atomic numbers near 56. The conversion of protons into neutrons to form the iron-peak elements that we observe, which almost always have more neutrons than protons, is then accomplished by the slow process of continuum electron capture. The time for the electron captures to occur is assumed limited by the various neutrino-loss mechanisms (Chiu and Stabler 1961). Fowler and Hoyle have shown that the predicted isotopic abundances in the iron peak depend in a crucial way on the number of electron captures that can occur in the time allowed by the neutrino processes, i.e., on the stellar electron-capture lifetimes.

The rate of stellar energy loss in the URCA process (Gamow and Schoenberg 1941; Chiu 1961) is determined by the electron-capture lifetimes of the nuclei involved. Colgate and White (1963) have proposed an interesting application of the URCA mechanism. Other astrophysical applications of stellar electron-capture lifetimes will no doubt be found as the interlocking studies of nuclear astrophysics and stellar models become more accurate and more detailed.

b) Outline of this Paper

The modern theory of nuclear beta-decay (see, e.g., Konopinski 1959) is applied in this paper to the calculation of continuum electron-capture rates for transitions of arbitrary degrees of forbiddenness. This work is a generalization of the author's previous studies of allowed continuum electron capture (Bahcall 1962 a,b); Schatzman (1958) has reviewed earlier work on allowed captures. The author is not aware of any previous investigations of forbidden stellar electron captures.

No derivations are given in this paper; the results that are presented were obtained by techniques developed for analyzing terrestrial beta-decay processes (Konopinski 1963 a,b). The equations that are most useful for astrophysical applications are discussed in detail. The nuclear physics uncertainties that are present in the prediction of the decay rates of certain classes of forbidden transitions are also described. Particular emphasis is placed upon using, whenever possible, terrestrial beta-decay measurements to supplement the theoretical stellar formulae.

In Section II, "exact" theoretical expressions are presented that give the rate of capture of a single continuum electron in a nuclear transition of any degree of forbiddenness. We also present the "normal approximations" to the "exact" expressions and examine the validity of these approximations for stellar-interior problems. The work in this section is based upon the analysis by Konopinski (1963 a,b) of closely related terrestrial beta-decay processes. In Section III, the results of Section II are generalized to describe the capture of electrons from a Fermi-Dirac gas of arbitrary temperature and density. We also introduce generalized phase-space functions that are convenient for predicting stellar rates when laboratory information regarding reactions (1), (3), or (4) is available. Sections IV-VI are devoted to a detailed explanation of how the formulae developed in Sections II and III can be used to predict stellar capture rates for stable isotopes and for isotopes that decay terrestrially by positron emission or electron capture. Some illustrative examples are treated in Section VII; they are: (a) the electron-capture lifetime of a proton, (b) the stellar beta-decay of K^{40}, and (c) the effect of forbidden transitions on equilibrium-process abundances.

The necessary information for predicting a specific decay rate can be obtained by reading the appropriate one of Sections IV-VI and referring occasionally to Sections II and III for definitions and remarks concerning the accuracy of the approximations.

II. CAPTURE OF A SINGLE CONTINUUM ELECTRON

The work in this section is based upon Konopinski's (1963 a,b) general treatment of positron emission and our notation is the same as his. In subsection (a), we present theoretical expressions, exact to second order in the weak coupling constant, for the rate of capture of a single continuum electron; these expressions apply to nuclear transitions of any degree of forbiddenness. The normal approximations to the exact expressions are given is subsection (b);

the validity of these normal approximations for stellar-interior problems is also discussed in subsection (b).

a) General Results

In order to calculate the rate of reaction (2), it is necessary to use for the initial electron state a Coulomb distorted plane wave that has an outgoing spherical wave. On the other hand, in calculations of terrestrial positron and electron emission rates, equations (3) and (4), a Coulomb distorted plane wave that has an incoming spherical wave is required for the final state (Breit and Bethe 1954).

Nuclear beta-decay interactions are represented by the following Hamiltonian density:

$$H_\beta = G2^{-1/2}\left[\bar{\psi}_\nu \,\gamma_\alpha(1+\gamma_5)\,\psi_e\right]\left[\bar{\psi}_n \,\gamma_\alpha(C_V - C_A\,\gamma_5)\,\psi_p\right] + \text{H.c.}, \qquad (5)$$

where all symbols have their usual meaning (Konopinski 1959; Bahcall 1962a). Expansion of ψ_e and ψ_ν in angular-momentum eigenstates leads, after an integration over electron directions and an average over electron polarizations, to the following equation for the transition probability for capture of an electron in the momentum interval $d^3\underset{\sim}{p}$:

$$d\lambda = \Sigma_{\kappa,\mu,\bar{\kappa},\bar{\mu}}\,\Sigma_{M_f}\,\Sigma_{M_i}^{av}\,\frac{(VGqp)^2}{(2\pi)^5}\,dp\,\left|\langle f|h_\beta(\kappa,\mu,\cdot\,\bar{\kappa},\bar{\mu})|i\rangle\right|^2 \qquad (6a)$$

where

$$h_\beta \equiv 2^{-1}\int d^3\underset{\sim}{r}\,(\bar{\psi}_n\,\gamma_\alpha(C_V - C_A\,\gamma_5)\,\psi_p)\,(\bar{\psi}_{\kappa,\mu}\,\gamma_\alpha(1+\gamma_5)\,\psi_{\bar{\kappa},\bar{\mu}}), \qquad (6b)$$

and

$$q \equiv W_o + W \qquad (6c)$$

Equations (6) are not changed if one requires that ψ_e have an incoming spherical wave instead of an outgoing spherical wave. The only physical fact required to prove this result is that stellar-interior electrons are unpolarized.

In equations (6), $\psi_{\kappa,\mu}$ is a Coulomb spherical wave for an electron and $\psi_{\bar{\kappa},\bar{\mu}}$ is a pure spherical wave for a neutrino (Rose 1961). Also, p is the magnitude of the electron's momentum and W is its total energy; q is the magnitude of the neutrino's momentum and W_o is the difference between initial and final <u>nuclear</u> masses. We have assumed one incident electron per volume V and have set $\hbar = m_e = c = 1$. The above choice of units is used throughout this paper except where explicitly stated otherwise.

Two convenient methods can be used to calculate the transition probability given by equation (6a): 1) direct expansion of h_β in vector spherical harmonics; 2) substitution, with appropriate modifications, of h_β^+ for the positron emission interaction treated by Konopinski (1963 <u>a</u>,<u>b</u>). Nuclear matrix elements obtained by method 2) refer to transitions from final to initial states and hence particular care must be taken in relating matrix elements that occur in 2) to the more usual ones that refer to transitions from initial to final states. In order to avoid mistakes in relative phase among the nuclear matrix elements, we have carried out the calculation of $d\lambda$ by both methods.

We find:

$$d\lambda = \frac{G^2}{4\pi^3} \, dp \, p^2 \, q^2 \, F(Z,W) \, S(W,Z) \quad , \tag{7}$$

where $S(W,Z)$ is the same shape factor that occurs in positron emission calculations (Konopinski 1963 <u>a</u>,<u>b</u>) except for the substitutions

$$[g_K(-Z)]_{\beta^+} \rightarrow [g_K(+Z)]_{\text{electron capture}} \tag{8a}$$

and

$$[f_K(-Z)]_{\beta^+} \rightarrow - [f_K(+Z)]_{\text{electron capture}} \quad . \tag{8b}$$

The quantity $F(Z,W)$ is the well-known Fermi function (Konopinski 1959; Bahcall 1962a). Note that equation (7) corresponds to a capture cross section for unpolarized, randomly directed electrons given by:

$$\bar{\sigma} = \frac{G^2 q^2 F S(W,Z)}{2\pi v} , \tag{9}$$

where v is the electron's velocity.

The theoretical shape factor for continuum electron capture has the following form:

$$S(W,Z) = \Sigma_{J,j,\bar{j}} \; S_{J\,j\,\bar{j}} \tag{9a}$$

where, in Konopinski's notation,

$$S_{J\,j\,\bar{j}} = (16\pi^2/1+\gamma_o)\,\rho_J^{\,2}\,R^{2j-1}\left[D_-^2\,L_{j-\frac{1}{2}} + D_+^2\,\bar{M}_{j-\frac{1}{2}} - 2\,D_-\,D_+\,\bar{N}_{j-\frac{1}{2}}\right] \tag{9b}$$

In equations (9), j, $\bar{j}$, and J are, respectively, the angular momentum of the captured electron, the angular momentum of the emitted neutrino, and the total lepton angular momentum; R is an "average" nuclear radius; L, $M = \bar{M}/R^2$, and $N = \bar{N}/R$ are combinations of electron radial waves introduced by Konopinski and Uhlenbeck (1941) in their original paper on forbidden beta decay. The quantities D_+ and D_- are combinations of neutrino radial waves with beta-moments and are defined by Konopinski (1963b); ρ_J and γ_o are numerical functions that are also defined in Appendix A.

Expression (9b) differs from the corresponding positron shape factor only in the sign of the $D_+ D_- \bar{N}_{j-\frac{1}{2}}$ term.

We shall only make use of equations (7) and (9) in the normal approximation to be described in the next subsection. It is useful, however, to have available the more general expressions since special cases (Konopinski 1963 a,b) can require more exact treatment than is afforded by the normal approximation.

b) Normal Approximations

(i) Validity of the Normal Approximation

The normal approximation consists of retaining only the leading terms in a power series expansion of S in terms of qR and pR. In transitions for which the terms independent of R give a nonvanishing capture rate, the normal approximation is equivalent to the usual allowed approximations that were used to derive the capture rate for allowed decays (Bahcall 1962a).

Two necessary criteria for the validity of the normal approximation for continuum electron capture in stars are:

$$R \ll \frac{1}{W_o + \langle W \rangle} \quad , \qquad (10a)$$

and

$$R \ll \frac{1}{\langle W \rangle} \quad , \qquad (10b)$$

where W_o is the difference between initial and final nuclear masses and $\langle W \rangle$ is some maximum effective total energy of the captured electron. Equations (10) follow from the requirement that qR and pR are small compared to unity. Note that in astrophysical applications W_o can be either positive or negative.

The criterion for the validity of the normal approximation in terrestrial electron and positron emission is equation (10a) with $\langle W \rangle$ replaced by -1. This criterion is easily fulfilled by every known case of terrestrial nuclear beta decay (Konopinski 1963b). Hence equation (10b) by itself can be used to test the validity of the normal approximation in stars. Substituting $1.2\ A^{1/3} \times 10^{-13}$ cm for R in equation (10b), we find that equation (10b) requires

$$T_{10} \ll 125/A^{1/3} \qquad (11a)$$

for nondegenerate electrons and

$$\rho_9 \ll 300/A \qquad (11b)$$

for completely degenerate electrons. Here, T_{10} is the stellar temperature in units of 10^{+10} ^oK and ρ_9 is the stellar density in units of 10^{+9} gm/cm^3. Equations (11) are well satisfied for all stellar situations in which nuclear physics studies have so far been made, but it is possible that equation (11b) is not satisfied in some white dwarf stars.

The normal approximation may also fail in cases in which there is an accidental cancellation among the nuclear-beta moments that occur in the lowest nonvanishing order of S. Such cancellations cannot be predicted theoretically on the basis of our present knowledge of nuclear structure, but they can frequently be detected by terrestrial measurements of shape factors and decay rates. Konopinski (1963b) has discussed the classes of decays for which failure of the normal approximation is most likely.

(ii) Normal Shape Factors

If only the leading terms in L, $\overline{M}$, and $\overline{N}$ are retained for the case of a field due to a point nucleus, equation (9b) becomes

$$S_{J\, j\, \bar{j}} = \frac{16\pi^2}{1+\gamma_o}\, \rho_J^{\,2}\, R^{2j-1}\, L_{k-1} \left[D_-^{\,2} + \frac{k-\gamma}{k+\gamma} D_+^{\,2} + 2\,\frac{D_- D_+ \alpha Z}{k+\gamma} \right] , \tag{12a}$$

where

$$k \equiv j + \frac{1}{2} \tag{12b}$$

and

$$\gamma \equiv (k^2 - \alpha^2 Z^2)^{\frac{1}{2}} . \tag{12c}$$

Making the approximation

$$k - \gamma \cong (\alpha\, Z)^2/2k \tag{13}$$

in equation (12a) and neglecting terms of order $(\alpha\ Z)^3$ or higher, we find that

$$S_{J\, j\, \bar{j}} \cong \frac{16\pi^2}{1+\gamma_o}\, \rho_J^{\,2}\, R^{2j-1}\, L_{k-1} \left[D_- + \frac{\alpha Z}{2k} D_+ \right]^2 . \tag{14}$$

Equation (14) has exactly the same form as the partial shape factor for positron emission.

The leading terms for D_+ and D_- are proportional to $R^{\bar{j}-\frac{1}{2}}$ and hence for $\Delta I = |I' - I| \geq 1$, where I is the initial nuclear spin and I' is the final nuclear spin, a single value of J dominates equation (9) in the normal approximation. This single dominant value of J is the minimum one defined by

$$\begin{aligned} J &= j + \bar{j} \\ &= \Delta I \quad . \end{aligned} \tag{15}$$

For <u>unique transitions</u>,

$$S_n^{(n+1)}\ (\Delta I \geq 1) \cong \frac{8\pi n!\ R^{2n}\ C_A^{\ 2}}{(1+\gamma_o)(2n+1)!!} \langle \sigma \cdot T^n_{n+1} \rangle^2\ \Sigma_{j,\bar{j}}\ A_J(j,\bar{j})\ L_{j-\frac{1}{2}}\ q^{2(n-j)+1}\ , \qquad (16)$$

where $n \equiv \Delta I - 1 = J - 1$ is the degree of forbiddenness and the product of initial and final nuclear parities satisfies $\pi_i \pi_f = (-1)^n$. The summation over j and $\bar{j}$ in equation (16) is restricted to j and $\bar{j}$ satisfying equation (15). The quantity $A_J(j,\bar{j})$ is defined by the following equation:

$$A_J(j,\bar{j}) = \frac{(2j)!!}{(2J-2j)!!\ (j-1/2)!\ (J-j-1/2)!}\ . \qquad (17)$$

The vector spherical harmonics, T^n_{n+1}, and the reduced nuclear matrix elements, $\langle \sigma \cdot T^n_{n+1} \rangle$, are defined in Appendix A. The phases of nuclear states have been chosen so that all nuclear matrix elements are real.

The notation for the shape-factor (16) is one already current in the literature; the subscript refers to the degree of forbiddenness and the superscript to the total angular momentum ejected. Unique decays, as their name implies, take place via a single nuclear matrix element.

The allowed $\Delta I = 1^+$ decays have a shape-factor that can be calculated from equation (16); we obtain:

$$\begin{aligned} S_o^{(1)} &= 4\pi\ C_A^{\ 2} \langle \sigma \cdot T_1^o \rangle^2 \\ &\equiv C_A^{\ 2} \langle \sigma \rangle^2\ . \end{aligned} \qquad (18)$$

The allowed $\Delta I = 0^+$ shape-factor cannot be calculated from equation (15) since for $\Delta I = 0$ the following possibility also leads to a shape-factor independent of R:

$$\begin{aligned} j - \bar{j} &\equiv 0 \\ &= \Delta I \qquad . \end{aligned} \tag{19}$$

Hence equation (15) is not satisfied. When equation (19) is used in equation (14), the contribution to the allowed shape-factor is found to be:

$$S_o^{(o)} = C_V^2 \langle 1 \rangle^2 \qquad . \tag{20}$$

The general shape-factor for allowed decays in the normal approximation is therefore:

$$\begin{aligned} S_{\text{allowed}} &= C_V^2 \langle 1 \rangle^2 + C_A^2 \langle \sigma \rangle^2 \\ &\equiv \xi \qquad . \end{aligned} \tag{21}$$

Equation (21) is in agreement with previous investigations of allowed continuum electron capture (Bahcall 1962a).

For $\Delta I \geq 1$ parity forbidden transitions,

$$S_{(n)}^{(n)} (\Delta I \geq 1) \cong \frac{8\pi (n-1)!\, R^{2(n-1)}}{(1+\gamma_o)(2n-1)!!} \Sigma_{j,\bar{j}}\, A_J(j,\bar{j})\, L_{j-\frac{1}{2}}\, q^{2(J-j)-1}\, M_j^{\,2}(J) \tag{22}$$

where $n = J = \Delta I$ is the degree of forbiddenness and $\pi_i \pi_f = (-1)^{\Delta I}$. Here $M_j(J)$ is the positron version of the parity-forbidden combination of nuclear matrix elements obtained by Konopinski (1963 a,b) and defined in Appendix A. The quantity $M_j(J)$ is a linear combination of three independent nuclear matrix elements and hence parity forbidden transitions are particularly susceptible to cancellation anomalies.

The first parity-forbidden transitions with $\Delta I = 1^-$ have a shape-factor

$$S_1^{(1)} \cong \left[C_V \langle \underset{\sim}{\alpha} \rangle - \frac{\alpha Z}{2} \left(C_A \langle \underset{\sim}{\sigma} \times \underset{\sim}{r} \rangle - C_V \langle i\hat{r} \rangle \right) \right]^2 \qquad (23)$$

that is independent of energy. The 0^+ to 0^- shape-factor,

$$S_1^{(o)} \cong C_A{}^2 \left[\langle \gamma_5 \rangle - \frac{\alpha Z}{2} \langle \underset{\sim}{\sigma} \cdot \hat{r} \rangle \right]^2 \quad , \qquad (24)$$

is also energy-independent. The general first parity-forbidden shape-factor is:

$$S_1 \cong S_1^{(o)} + S_1^{(1)} \quad . \qquad (25)$$

III. STELLAR CAPTURE RATES

a) Capture in a Fermi Gas

Equation (7) gives the capture rate of a single continuum electron with a definite total energy W. In order to calculate stellar capture rates, we must multiply equation (7) by the probability that a stellar electron actually has the energy W; this probability is essentially determined by the Fermi-Dirac distribution function,

$$FD \equiv (1 + e^{-\nu + W/kT})^{-1} \quad . \qquad (26)$$

If the correct distribution function for a specific problem departs significantly from the Fermi-Dirac function, then the formulae given below should be used with $(1 + e^{-\nu + W/kT})^{-1}$ replaced by the correct distribution function.

The notation in equation (26) is the same as that used previously by ter Haar (1956) and by the present author (Bahcall 1962 a,b). For nondegenerate electrons, convenient analytic and numerical expressions for ν have been given by several authors (Chandrasekhar 1957; Chiu 1961; Fowler and Hoyle 1963). Note that our ν is equal to $(m_e c^2/kT - \alpha)$ in Chandrasekhar's notation, μ/kT in Chiu's notation, and $m_e c^2 \varphi/kT$ in Fowler and Hoyle's notation. For highly degenerate electrons, ν is approximately equal to the Fermi energy divided by kT (Bahcall 1962c).

Multiplying equation (7) by the Fermi-Dirac distribution function, we obtain:

$$\lambda = \frac{G^2 \quad \langle S \rangle}{2\pi^3 \; (\hbar/mc)^7 \; m^2 c^3} \quad , \tag{27a}$$

where the dimensionless average shape factor is defined by the equation

$$\langle S \rangle \equiv \int_{P_o}^{\infty} dp \; p^2 \; q^2 \; F(Z,W) \; S(W,Z) \; (1 + e^{-\nu + W/kT})^{-1} \quad . \tag{27b}$$

The threshold momentum, P_o, is defined in terms of the difference between initial and final nuclear masses, W_o; the relation is:

$$P_o = \begin{cases} 0, \text{ if } W_o \geq -m_e c^2 & \text{(exoergic capture).} \\ (W_o^2 - 1)^{1/2}, \text{ if } W_o \leq -m_e c^2 & \text{(endoergic capture).} \end{cases} \tag{27c}$$

An extra factor of two occurs in equation (27a) because two electrons can be in the same momentum eigenstate if their spin projections are different.

b) Generalized Phase-Space Functions

For unique decays, it is convenient to introduce dimensionless generalized phase-space functions, $K_n^{(n+1)}$, that are defined by the following equation:

$$\langle S_n^{(n+1)} \rangle \equiv 4\pi \left[\frac{R^n\, C_A \langle \sigma \cdot T_{n+1}^n \rangle}{(2n+1)!!} \right]^2 K_n^{(n+1)} \tag{28}$$

Here n is still the degree of forbiddenness and $\Delta I = n + 1$. Equations (16), (27), and (28) imply that

$$K_n^{(n+1)} \equiv \frac{2(n!)(2n+1)!!}{(1+\gamma_o)} \Sigma_{j,\bar{j}}\, A_J(j,\bar{j}) \int_{P_o}^{\infty} dp\, p^2\, F(Z,W)\, L_{j-\frac{1}{2}}\, q^{2(n-j)+1} \left(1+ e^{-\nu+W/kT}\right)^{-1} , \tag{29}$$

where $j + \bar{j} = n + 1$. If terms of order $(\alpha Z)^2$ in $L_{j-\frac{1}{2}}$ are neglected, then $K_n^{(n+1)}$ has the following symmetrical form:

$$K_n^{(n+1)} \cong (2n+1)!\, \Sigma_{j,\bar{j}} \frac{1}{(2j)!\,(2\bar{j})!} \int_{P_o}^{\infty} dp\, p^2\, q^2\, F(Z,W)\, p^{2j-1}\, q^{2\bar{j}-1} \left(1+ e^{-\nu+W/kT}\right)^{-1} . \tag{30}$$

Equation (30) is sufficiently accurate for most astrophysical applications.

Recall that the quantity $(f\tau)$, which is used to characterize allowed terrestrial beta decays, satisfies the relation

$$(f\, \tau_{\frac{1}{2}})_{lab}\, \xi \equiv \frac{2\pi^3 \ln 2}{G^2} \left(\frac{\hbar}{mc}\right)^7 m^2 c^3 \quad . \tag{31a}$$

The Coulomb-corrected phase-space function, f, is defined by the equation

$$f\,(\pm Z,W) \equiv \int_o^{P_{max}} dp\, p^2\, q^2\, F(\pm Z,W) , \tag{31b}$$

where by convention the plus sign applies for electron emission and the minus sign for positron emission. The maximum electron or positron momentum in a terrestrial decay is here denoted by P_{max}. The value of $(f\tau)_{lab}$ is conventionally used to classify terrestrial beta decays since it corrects the half-life for the "accidental" effects of energy release and nuclear charge.

For allowed stellar captures,

$$K_0^{(1)} = \int_{P_0}^{\infty} dp\ p^2\ q^2\ F(Z,W)\ (1 + e^{-\nu + W/kT})^{-1} \quad , \tag{32a}$$

$$\equiv K \ , \tag{32b}$$

where K is the integral over the available phase-space, with Coulomb and statistical corrections, that was introduced in previous studies of allowed stellar captures (Bahcall 1962 a,b). Note the similarity between the definition of f, equation (31b), and the definition of K, equation (32a).

The stellar phase-space function K satisfies, according to equations (21), (27) and (28), the relation:

$$(K\ \tau_{\frac{1}{2}})_{star}\ \xi = \frac{2\pi^3\ \ln 2}{G^2} \left(\frac{\hbar}{mc}\right)^7 m^2 c^3 \quad , \tag{33}$$

which is entirely analogous to the relation, equation (31a), satisfied by the terrestrial phase-space function f. Thus K is a natural quantity to use in characterizing allowed stellar captures. We shall also see, in the following sections, that by focusing attention on the product $(K\ \tau_{\frac{1}{2}})_{star}$ one can utilize laboratory beta-decay measurements to make simple and accurate predictions of allowed stellar capture rates.

First parity-forbidden transitions have shape-factors that, in the normal approximation, are energy independent combinations of nuclear matrix elements; the explicit definitions of the first parity-forbidden shape-factors were given in equations (23)-(25). Because these shape-factors are energy independent, there exist relations for first parity-forbidden transitions that are entirely analogous to relations (31) and (33) for allowed decays. The relations for first parity-forbidden decays are:

$$(f\,\tau_{\frac{1}{2}})_{lab}\, S_1 = 2\pi^2\, \ell n\, 2/G^2 \qquad , \tag{34}$$

and

$$(K\,\tau_{\frac{1}{2}})_{star}\, S_1 = 2\pi^2\, \ell n\, 2/G^2 \qquad , \tag{35}$$

where S_1 is the particular combination of nuclear matrix elements defined by equation (25).

Thus K is also the appropriate quantity to use in characterizing first parity-forbidden stellar captures.

IV. STELLAR CAPTURE RATES FOR TERRESTRIAL POSITRON EMITTERS

In this section, we show how stellar capture rates can be calculated from the results of laboratory studies of terrestrial positron decays.

Since positron decay is assumed to occur on earth, continuum electron capture is exoergic and P_o is set equal to zero throughout this section.

a) Allowed and First Parity-Forbidden Transitions

It has been shown previously (Bahcall 1962 a,b) that continuum electron-capture lifetimes for nuclei that decay terrestrially by allowed positron emission can be calculated from the following relation:

$$(\tau_{\frac{1}{2}})_{star} = (f\,\tau_{\frac{1}{2}})_{lab}\; K^{-1} \quad . \tag{36}$$

Here, $(f\,\tau_{\frac{1}{2}})_{lab}$ is the measured positron ft-value and K is the stellar phase-space function defined by equation (32).

For allowed transitions, equation (36) follows immediately from equations (31) and (33). Equation (36) is also valid, in the normal approximation, for first parity-forbidden transitions; the proof of equation (36) for first parity-forbidden transitions follows immediately from equations (34) and (35).

The simple physical fact expressed by equation (36) is that the rate of an allowed (or first parity-forbidden) beta-decay transition, on earth or in a star, is proportional to the total available phase-space.

Equation (36) is a powerful relation since it is independent of nuclear matrix elements, nuclear radii, and beta-decay coupling constants; all these quantities are ones that cannot be reliably predicted with our present knowledge of nuclear structure. Note that equation (36) can be used to estimate stellar decay rates even if the corresponding terrestrial positron decay has not been thoroughly investigated. This is because considerable experimental information exists that correlates observed $(f\,\tau_{\frac{1}{2}})_{lab}$ values with models of nuclear structure (Feenberg 1955; Mayer and Jensen 1955; Konopinski 1963b), and thus it is often possible to guess fairly accurately a value of $(f\,\tau_{\frac{1}{2}})_{lab}$ to be inserted in equation (36).

The fact that equation (36) can be used to estimate stellar decay rates even when the corresponding laboratory decay rates have not been measured is a particularly useful feature in studies of stars at very high temperatures. This feature is useful because the beta decay of excited nuclear states, a process which cannot easily be studied in the laboratory, frequently dominates the stellar-interior beta decay of nuclei at temperatures in excess of 10^{+8} ^{o}K (Cameron 1959b; Bahcall 1962c; Fowler and Hoyle 1963).

Approximate analytic expressions for K were given by Bahcall (1962 a,b) for both degenerate and nondegenerate electrons. Fowler and Hoyle (1963) have since developed analytic approximations for K that are particularly convenient for the conditions thought to obtain in stars during the formation of the iron-peak elements.

b) Unique Transitions

The transition probability for unique positron emission on earth is (Konopinski 1963 a,b):

$$\lambda_{lab} = \frac{G^2}{2\pi^3} \int_o^{P_{max}} dp\, p^2\, F(-Z,W)\, (W_o - W)^2\, S_{n+1}^{(n)}\, (Z,W) \quad , \qquad (37)$$

where $S_{n+1}^{(n)}$ has, in the normal approximation, the same form, equation (16), for positron emission as for continuum electron capture. Recall that unique n^{th}-forbidden transitions have $\Delta I = n + 1 = J$ with $\pi_i \pi_f = (-1)^n$.

One can again obtain an equation that is independent of nuclear radii, nuclear matrix elements, and other uncertain parameters by combining equations (16), (27) and (37), we find:

$$(\tau_{\frac{1}{2}})_{star} = (f_n^{(n+1)}\, \tau_{\frac{1}{2}})_{lab}\, (K_n^{(n+1)})^{-1} \quad , \qquad (38)$$

where, analogous to equation (29),

$$f_n^{(n+1)} = \frac{2(n!)(2n+1)!!}{1+\gamma_o} \Sigma_{j,\bar{j}}\, A_J(j,\bar{j}) \int_0^{P_{max}} dp\, p^2\, F(Z,W)\, L_{j-\frac{1}{2}}\, q^{2(n-j)+1} \qquad (39a)$$

$$\cong (2n+1)!\, \Sigma_{j,\bar{j}} \frac{1}{(2j)!\,(2\bar{j})!} \int_0^{P_{max}} dp\, p^2\, q^2\, F(Z,W)\, p^{2j-1}\, q^{2\bar{j}-1}\ . \qquad (39b)$$

Notice that for allowed decays $f_o^{(1)}$ reduces to the usual Coulomb-corrected phase-space factor f.

Equation (38) states that the rate of a unique transition, on earth or in a star, is proportional to the generalized phase-space available.

The quantity $f_n^{(n+1)}$ can easily be computed if a laboratory measurement of the maximum positron momentum, P_{max}, has been performed. The maximum momentum P_{max} has, in fact, been measured for almost all known terrestrial positron emitters. Thus equation (38), like equation (36), expresses the stellar capture rate in terms of the most readily measured nuclear parameters.

Equation (38) can also be used to estimate stellar capture rates of isotopes whose terrestrial decay has not been thoroughly investigated in the laboratory, since there also exists a considerable amount of experimental information that correlates $(f_n^{(n+1)}\ \tau_{\frac{1}{2}})_{lab}$ with models of nuclear structure (Feenberg 1955; Mayer and Jensen 1955; Konopinski 1963b).

c) General Parity-Forbidden Transitions

The transition probability for terrestrial parity-forbidden positron emission is (Konopinski 1963 a,b):

$$\lambda_{lab} = \frac{G^2}{2\pi^3} \int_0^{P_{max}} dp\, p^2\, F(-Z,W)\, (W_o - W)^2\, S_n^{(n)}\, (Z,W) \quad , \qquad (40)$$

where $S_n^{(n)}$ has, in the normal approximation, the same form for positron emission as for continuum electron capture (see equation (22)). Recall that $n^{\underline{th}}$ parity-forbidden transitions have $\Delta I = n = J$ with $\pi_i \pi_f = (-1)^n$.

It is in general not possible to derive for parity-forbidden captures an equation, similar to equation (38), from which stellar capture rates can be calculated when only P_{max} and $(\tau_{\frac{1}{2}})_{lab}$ are known. This is because the shape factor, $S_n^{(n)}$, for parity-forbidden transitions does not have a unique energy dependence; the energy dependence of parity-forbidden transitions is determined by a collection of nuclear matrix elements that are, in general, unknown. The complicated combinations of nuclear matrix elements that determine the energy dependence of $S_n^{(n)}$ were introduced as parameters called $M_j(J)$ in equation (22) and are defined in Appendix A. We were able to obtain a simple equation, equation (36), for first parity-forbidden continuum captures only because $S_1^{(0)}$ and $S_1^{(1)}$ are independent of energy in the normal approximation.

One can, however, derive an equation for general parity-forbidden transitions that can be used to calculate stellar capture rates if the terrestrial positron shape factor, $S_n^{(n)}$, the maximum positron momentum P_{max}, and the laboratory positron half-life, $(\tau_{\frac{1}{2}})_{lab}$, have all been determined experimentally. We find, analogous to equations (38) and (39),

$$(\tau_{\frac{1}{2}})_{star} = (f_n^{(n)} \tau_{\frac{1}{2}})_{lab} \left(K_n^{(n)}\right)^{-1} , \tag{41a}$$

where now

$$f_n^{(n)} \equiv \frac{2(n!)(2n+1)!!}{1+\gamma_o} \Sigma_{j,\bar{j}} A_n(j,\bar{j}) \int_0^{P_{max}} dp\, p^2\, F(-Z,W)\, L_{j-\frac{1}{2}}\, q^{2(n-j)-1} M_j{}^2(n) , \tag{41b}$$

and

$$K_n^{(n)} \equiv \frac{2(n!)(2n+1)!!}{1+\gamma_o} \Sigma_{j,\bar{j}} A_n(j,\bar{j}) \int_{P_o}^{\infty} dp\, p^2\, F(Z,W)\, L_{j-\frac{1}{2}}\, q^{2(n-j)-1}\, M_j^{\,2}(n) \times (1 + e^{-\nu+W/kT})^{-1} \qquad (41c)$$

depend on the relevant nuclear matrix elements.

Equations (41) can be reliably used to predict stellar capture rates only if the laboratory positron shape-factor, i.e., the unknown part of the integrand of equation (41b), has been measured over a range of momenta that significantly overlaps the important range of momenta that occurs in equation (41c). For most stellar situations, the two ranges of momenta do overlap enough to make reliable predictions possible.

If the terrestrial positron spectrum has not been measured, one can still make very rough estimates of the stellar capture rate by guessing a form for $S_n^{(n)}(Z,W)$ and then applying equations (41). This procedure is very inaccurate for parity-forbidden transitions, since they are likely to exhibit anomalies due to cancellations among nuclear matrix elements.

V. STELLAR CAPTURE RATES VERSUS TERRESTRIAL CAPTURE RATES

In this section, we discuss how one can predict the rate of stellar continuum electron capture for an isotope that decays terrestrially by capture of a bound atomic electron. The threshold momentum, P_o, is again zero.

The terrestrial captures of primary interest are ones involving K-shell electrons since capture from higher shells usually accounts for a small fraction, of the order of 10 per cent, of the total decay rate. Moreover, the theoretical corrections for captures from higher shells can now be made with considerable accuracy (Bahcall 1962d, 1963). Hence we can regard measured total capture rates as providing accurate values for K-capture rates.

a) Allowed and First Parity-Forbidden Transitions

(i) General Results

It is convenient to define a quantity, $f_{e.c.}$, by the following equation:

$$f_{e.c.}(Z,W_o) = 2\pi^2 \Sigma_n q_{ns}^2(W_o) |\psi_{ns}(0)|^2 , \qquad (42)$$

where the sum extends over all atomic principal quantum numbers, n. In equation (42), $\psi_{ns}(0)$ is the value of an ns-electron's wave function evaluated at the nucleus and q_{ns} is the momentum of a neutrino emitted when an ns-electron is captured by the nucleus. Note that

$$q_{ns} \cong W_o + 1 - b(ns) , \qquad (43)$$

where $b(ns)$ is the positive binding energy of an ns-electron in the initial atom.

Using well-known results (Brysk and Rose 1960; Konopinski 1963 a,b) for bound electron capture, one can easily show that for allowed decays:

$$(f_{e.c.}\,\tau_{\frac{1}{2}})_{lab}\,\xi = G^{-2}(2\pi^3 \ln 2) . \qquad (44)$$

and for first parity-forbidden decays

$$(f_{e.c.}\,\tau_{\frac{1}{2}})_{lab}\,S_1 = G^{-2}(2\pi^3 \ln 2) . \qquad (45)$$

Capture from other than s-orbits is neglected in equations (44) and (45).

Equations (44) and (45) have exactly the same form as equations (31) and

(33) and hence the relation

$$(\tau_{\frac{1}{2}})_{star} = (f_{e.c.}\ \tau_{\frac{1}{2}})_{lab}\ K^{-1} \tag{46}$$

obtains for allowed and first parity-forbidden electron captures. Note that equation (46), like equation (36), is independent of nuclear matrix elements, nuclear radii, G, C_V, and C_A.

Equations (44) and (45) also show that $f_{e.c.}$ plays the same role in electron capture as $f(-Z, W_o)$ plays in positron emission. Hence the empirical correlation that exists between $(f(-Z, W_o)\ \tau_{\frac{1}{2}})_{lab}$ and nuclear structure also provides an empirical correlation between $(f_{e.c.}\ \tau_{\frac{1}{2}})$ and nuclear structure, and equation (46), like equation (36), can be used to estimate a stellar capture lifetime even if the terrestrial decay has not been thoroughly investigated. One simply makes an educated guess for $(f\tau)_{lab}$ and inserts the guessed value in equation (46).

(ii) Approximate Expressions and Physical Interpretation

It is useful to derive some approximate relations for K and $f_{e.c.}$ in order to understand the physical interpretation of equation (46). If one assumes that $W_o + 1 \gg \langle E \rangle$, where $\langle W \rangle$ is some average electron kinetic energy in a star, then one can easily show that

$$K \cong \pi^2 n_e \langle F \rangle (W_o + 1)^2 \quad , \tag{47}$$

where n_e is the electron concentration and $\langle F \rangle$ is the Fermi function evaluated at $W = 1 + \langle E \rangle$. Equation (47) is valid for both degenerate and nondegenerate matter. For nondegenerate electrons, $\langle E \rangle \sim kT$ and, for degenerate electrons, $\langle E \rangle \sim E_F$, where E_F is the electron Fermi energy.

The quantity $f_{e.c.}$ also has a simple approximate form:

$$f_{e.c.} \cong 2\pi^2 (W_o + 1)^2 |\psi_{1s}(0)|^2 \tag{48a}$$

$$\cong 2\pi (W_o + 1)^2 (\alpha Z)^3 \quad . \tag{48b}$$

Thus:

$$\frac{(\tau_{\frac{1}{2}})_{star}}{(\tau_{\frac{1}{2}})_{lab}} \cong \frac{2 |\psi_{1s}(0)|^2}{n_e \langle F \rangle} \quad , \tag{49}$$

or:

$$\frac{(\tau_{\frac{1}{2}})_{star}}{(\tau_{\frac{1}{2}})_{lab}} \cong \frac{10 \mu_e Z^3}{\rho \langle F \rangle} \quad , \tag{50}$$

where ρ is the density in gm/cm^3 and μ_e is the mean molecular weight per electron.

Equation (49) shows that the ratio of the stellar to the laboratory lifetime is roughly equal to the ratio of the electron density at the nucleus for the laboratory and stellar situations, i.e., to the relative probability of finding an electron at the nucleus where it can be captured.

Equation (50) can be used to calculate quickly the order of magnitude of a stellar capture lifetime when the terrestrial capture lifetime is known.

b) Unique Captures

(i) General Results

For all unique decays, the rate at which two atomic ns-electrons are captured is (Konopinski 1963b):

$$\lambda_{ns} = |\psi_{ns}(0)|^2 \left[\frac{2G\, q_K\, C_A \langle \sigma \cdot T_{\Delta I}^{\Delta I-1} \rangle (q_K R)^{\Delta I-1}}{(2\Delta I - 1)!!} \right]^2 , \qquad (51)$$

where $\Delta I - 1$ is the degree of forbiddenness. Hence,

$$(\tau_{\frac{1}{2}})_{star} = (f_{e.c.}{}^{(\Delta I)}_{\Delta I-1}\, \tau_{\frac{1}{2}})_{lab} \left(K^{(\Delta I)}_{\Delta I-1} \right)^{-1} , \qquad (52a)$$

where

$$f_{e.c.}{}^{(\Delta I)}_{(\Delta I-1)} \equiv 2\pi^2\, \Sigma_n\, q_{ns}{}^{2\Delta I}\, |\psi_{ns}(0)|^2 . \qquad (52b)$$

Equation (52a) has the same structure as equation (38) and hence equation (52a) can also be used to estimate the stellar lifetime of isotopes whose laboratory decay has not been thoroughly investigated.

One can also prove that equations (49) and (50) are approximately satisfied for all unique transitions if $W_o + 1 \gg \langle E \rangle$. From equations (30) and (32), we find that

$$K^{(\Delta I)}_{\Delta I-1} \cong \langle |\underset{\sim}{p} + \underset{\sim}{q}|^{2(\Delta I-1)} \rangle K \qquad (53)$$

and therefore:

$$K^{(\Delta I)}_{\Delta I-1} \cong \pi^2\, q^{2\Delta I}\, n_e \langle F \rangle . \qquad (54)$$

In equation (53), $\langle |\underset{\sim}{p} + \underset{\sim}{q}|^{2(\Delta I-1)} \rangle$ is the average over all directions and energies of $(\underset{\sim}{p} + \underset{\sim}{q})^{2(\Delta I-1)}$. Equations (52) and (54), when combined, yield

$$\frac{(\tau_{\frac{1}{2}})_{star}}{(\tau_{\frac{1}{2}})_{lab}} \cong \frac{2\, |\psi_{1s}(0)|^2}{n_e \langle F \rangle}$$

as asserted.

c) General Parity-Forbidden Captures

Since the bound capture rate is dominated by K-capture, only a single combination of nuclear matrix elements, $M_{\frac{1}{2}}(\Delta I)$, enters in an important way the terrestrial capture formula for general parity-forbidden transitions. Unfortunately, the stellar capture rate depends on all $M_j(\Delta I)$ for which j and $\bar{j}$ satisfy equation (15). Thus for general parity-forbidden transitions one cannot derive an expression for the stellar capture rate that is independent of unknown nuclear quantities. Instead, one must use directly equations (22) and (27) with rough guesses for the unknown nuclear matrix elements and other nuclear quantities; this procedure is very inaccurate with our present knowledge of nuclear structure.

VI. INDUCED ELECTRON CAPTURE

In this section, we discuss the rate of continuum electron capture for an isotope that is stable on earth. Since the terrestrial reactions (1) and (3) do not occur, the relevant terrestrial decay is described by reaction (4),

$$(Z-1,A) \rightarrow (Z,A) + e^- + \bar{\nu} \quad .$$

The threshold momentum for stellar electron capture by the isotope (Z,A) is obviously not zero if reaction (4) occurs on earth.

a) Allowed and Unique Decays

The stellar capture lifetimes can again be computed from equation (36) for allowed captures and from equations (38) and (39) for all unique decays. The only change that is necessary in the formulae referring to laboratory decays is that $F(-Z,W)$ must be replaced by $F(+Z,W)$ in the expressions for $f(Z,W_o)$

and $f_n^{(n+1)}$ (Z,W_o). For allowed captures, the above substitution merely replaces $f(-Z,W_o)$ by the equally well-investigated function $f(+Z,W_o)$.

The stellar phase-space functions K and $K_n^{(n+1)}$ must now be calculated for P_o different from zero. Fowler and Hoyle (1963) have derived convenient analytic expressions for K, with nonzero P_o, that are valid for the conditions thought to obtain in the interiors of stars during the formation of the iron-peak isotopes. Their results can readily be extended to the more general functions, $K_n^{(n+1)}$, although the general results are very complicated.

Some analytic approximations for first-forbidden unique decays are given in Appendix B.

b) Parity-Forbidden Decays

The combinations of nuclear matrix elements that occur in theoretical expressions for the stellar capture rates are different from the combinations that occur in the expressions for terrestrial electron-emission rates. This is because the stellar capture probability is determined by the positron shape-factor and the terrestrial decay probability is determined by the electron shape-factor; the parity-forbidden shape-factors are well-known (Preston 1962; Konopinski 1963 a,b) to be linear combinations of reduced matrix elements with some coefficients that change sign when an electron is replaced by a positron. For example, the term $+ \alpha Z\ C_v \langle i\,\hat{r}\rangle/2$ which occurs in the first parity-forbidden shape-factor, $S_1^{(1)}$ of equation (23), for electron capture and positron emission, becomes $- \alpha Z\ C_v \langle i\,\hat{r}\rangle/2$ for electron emission; all other terms in $S_1^{(1)}$ are unchanged.

It is therefore not possible to calculate induced stellar parity-forbidden capture rates from the results of simple experiments on the related exoergic terrestrial decays. In a few cases, all the nuclear matrix elements have been determined, at least approximately, by a series of experiments and one can therefore use equations (22)-(25) and (27) to calculate directly the stellar capture lifetimes. In most cases, however, the best one can do is to make some crude guess for the relevant matrix elements and this will in general lead to large uncertainties in the computed lifetime.

VII. APPLICATIONS

In this section, we illustrate the results of the previous sections by calculating some stellar capture rates.

a) $e^- + p \rightarrow n + \nu$

(i) Stellar Rates

The electron-capture lifetime of a stellar proton is:

$$(\tau_{\frac{1}{2}})_{\text{star}} = (1175 \pm 30) \text{ sec } K^{-1} \quad . \tag{55}$$

To obtain equation (55), we have used, in equation (36), the value given by Durand _et al._ (1963) of $(f\,\tau_{\frac{1}{2}})_{\text{lab}}$ for the beta decay of the neutron.

For nondegenerate electrons,

$$K_{N-D} \cong e^{+\nu} \int_{|W_o|}^{\infty} dp\, p^2\, F(-|W_o| + W)^2\, e^{-\beta W} \tag{56}$$

$$\cong 2\beta^{-5}\, e^{+\nu-x}\, [x^2 + 6x + 12] \qquad ,$$

where

$$\beta \equiv m_e c^2/kT \tag{57a}$$

and

$$x \equiv \frac{|W_o|}{kT} \tag{57b}$$

$$\cong 2.54\ \beta \quad .$$

Note that the proton lifetime depends exponentially on the threshold energy divided by kT.

It is interesting to know the proton lifetime under the conditions thought to obtain in white dwarfs or in the early stages of an expanding universe. In both these situations, the electrons are relativistically degenerate and the appropriate function K is:

$$K_D = \frac{(W_F - |W_o|)^5}{5} + \frac{|W_o|}{2}(W_F - |W_o|)^4 + \frac{W_o^{\ 2}(W_F - |W_o|)^3}{3}, \tag{58}$$

where the total Fermi energy is defined by

$$W_F = (1 + (3\pi^2 n_e)^{2/3})^{\frac{1}{2}} \tag{59a}$$

$$= 511 \left[1.02 \times 10^{-4}\left(\frac{\rho}{\mu_e}\right)^{2/3} + 1\right]^{\frac{1}{2}} \text{kev.} \tag{59b}$$

In equation (59b), ρ is in gm/cm^3.

We have neglected Coulomb corrections in equations (56) and (58); this neglect gives rise to a maximum error of 6 per cent, which would occur for Fermi energies only slightly greater than $|W_o|$.

(ii) The Saxon Experiment

Saxon (1949) tried to observe reaction (a) with 1 Mev electrons from a van de Graaff accelerator, but was only able to set an upper limit of 2.5×10^{-35} cm^2 on the effective cross section. We can easily see that the expected theoretical cross section for reaction (a) is much less than the observed upper limit. Equations (7) and (31a) can be used to show that

$$\sigma_{e^- + H} = \pi^2 \left(\frac{\hbar}{mc}\right)^3 \left(\frac{W - |W_o|}{mc^2}\right)^2 \frac{(\ln 2)\, F(Z,W)}{v(f\,\tau_{\frac{1}{2}})_{lab}} \quad , \qquad (60)$$

where v is the velocity of the incoming electrons and $(f\,\tau_{\frac{1}{2}})_{lab}$ is the neutron ft-value. The theoretical cross section for 1 Mev electrons is, according to equation (60), only 3×10^{-45} cm^2.

b) K^{40} Decay

In order to understand the formation processes of A^{40} and Ca^{40}, it is necessary to have at least a semi-quantitative understanding of the beta-decay lifetime and branching ratios of K^{40} as a function of temperature and density. Bashkin (1962) has suggested that the decay of the 29 kev excited state of K^{40} may dominate its stellar beta decay. The astrophysically important levels for A^{40}, K^{40}, and Ca^{40} are shown in Figure 1.

The total K^{40} decay rate can be written as a sum of two terms,

$$\lambda_{star}^{total} = \lambda_{star}\,(K^{40} \rightarrow Ca^{40}) + \lambda_{star}\,(K^{40} \rightarrow A^{40}) \quad , \qquad (61a)$$

where

$$\lambda_{star}\,(K^{40} \rightarrow Ca^{40}) = \lambda_{\beta^-}\,(4^- \rightarrow 0^+) + e^{-3.5/T_8}\,\lambda_{\beta^-}\,(3^- \rightarrow 0^+) \ , \qquad (61b)$$

and

$$\lambda_{star}\,(K^{40} \to A^{40}) = \lambda_{e.c.}\,(4^- \to 2^+) + e^{-3.5/T_8}\left(\lambda_{\beta^+}\,(3^- \to 0^+) + \lambda_{e.c.}\,(3^- \to 2^+)\right) . \tag{61c}$$

In equations (61), nuclear energy levels are identified by their spins and parities and the temperature, T_8, is measured in units of 10^{+8} ^{o}K.

The stellar decay rate of the 1.321 Mev β^- transition of K^{40} to Ca^{40} will probably not be significantly different from its terrestrial value; this conclusion is based upon previous studies of the stellar decay rates of β^--emitters (Bahcall 1961, 1962a). The excited-state $3^- \to 0^+$ transition will probably not be appreciably faster than the ground state to ground-state decay. Thus we conclude that

$$\lambda_{star}\,(K^{40} \to Ca^{40}) \cong \lambda_{earth}\,(K^{40} \to Ca^{40}) \quad . \tag{62}$$

It is unlikely that the third-forbidden 3^- to 0^+ K^{40} to A^{40} transition will proceed as rapidly as the first-forbidden 3^- to 2^+ transition, since the two extra orders of forbiddenness would normally be far more important than the additional energy available for the more forbidden transition. Hence we expect that

$$\lambda_{star}\,(K^{40} \to A^{40}) = \lambda_{e.c.}\,(4^- \to 2^+) + e^{-3.5/T_8}\,\lambda_{e.c.}\,(3^- \to 2^+) \quad . \tag{63}$$

The transition probability, $\lambda_{e.c.}\,(4^- \to 2^+)$, can be calculated for any specified stellar temperature and density with the formulae given in Appendix B. The excited-state transition probability, $\lambda_{e.c.}\,(3^- \to 2^+)$, cannot be accurately predicted since no measurement of the excited-state decay rate has been made. A rough estimate of this decay rate can be made, however, by using equation (47) and assuming that $(f_{e.c.}\ \tau_{\frac{1}{2}})_{lab}$ is approximately $10^{+7.5 \pm 1.0}$ sec. The beta-decay branch, and lifetime, that dominate the stellar decay of K^{40} will vary with temperature and density approximately according to equations (62) and (63).

c) e-Process Capture Rates

Fowler and Hoyle (1963) have shown that the beta-decay rates of six isotopes, $Fe^{54,55,56}$ and $Ni^{56,57,58}$, determine, on their picture of the equilibrium process, the observed relative abundances of the elements in the iron peak. Since several of the ground-state to ground-state electron captures for these isotopes are forbidden, it is important to know if any of the forbidden transitions occur appreciably more rapidly than the allowed excited-state transitions that can also occur.

We have examined the most likely decay schemes of each of the above listed six isotopes in an attempt to determine if forbidden captures can greatly reduce the total decay rate predicted (Fowler and Hoyle 1963) by considering only allowed decays. In this investigation, we have adopted the standard e-process conditions of Fowler and Hoyle. The most important condition, for our examination, is that Fowler and Hoyle use an effective e-process temperature of $3.8 \times 10^{+9}\ ^{\circ}K$. This temperature corresponds to a Boltzman population factor for excited nuclear states of approximately e^{-3E}, where E is the excitation energy in Mev. Thus nuclear states with excitation energies as high as 1.5 Mev have an average population greater than or of the order of 1 per cent. Beta-decay experiments show that as a rough rule each successive degree of forbiddenness increases the appropriate ft-value by a factor of the order of 100 (Feenberg 1955; Konopinski 1963b).

Using the above facts, we have found that none of the six isotopes of interest are likely to have their beta-decay rates affected significantly by forbidden decays.

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APPENDIX A.

The notation and definitions in this appendix are taken from Konopinski (1963 a,b).

The vector spherical harmonics, $\underset{\sim}{T}^{L}_{Jm}$, are defined in terms of the spherical harmonics, Y_{Lm} , by the relation:

$$\underset{\sim}{T}^{L}_{Jm}(\underset{\sim}{r}) = \Sigma_{\mu=o,\pm 1} \langle L(m-\mu)\ 1(\mu)|J(m)\rangle\, Y_{Lm}\, \hat{e}_{\mu} \quad , \tag{A1}$$

where $\hat{e}_{o} = \hat{e}_{z}$, $\hat{e}_{\pm 1} = \mp 2^{-\frac{1}{2}} (\hat{e}_{x} \pm i\, \hat{e}_{y})$ when $\hat{e}_{x,y,z}$ are unit vectors in the directions of the Cartesian axes.

The nuclear beta-moments are introduced with the help of the Wigner-Eckart theorem. Let (V.A.) stand for the vector-addition coefficient $\langle I_f(M_f)\ J(m)|I_i(M_i)\rangle$, where I_i, I_f, M_i, M_f are the initial and final nuclear spin and spin projection, respectively. Then:

$$\langle I_f\ (M_f)|[i^J\ Y_{Jm}]^*|I_i\ (M_i)\rangle \equiv (V.A.)\ \langle Y_J\rangle \quad , \tag{A2}$$

$$\langle I_f\ (M_f)|\ \gamma_5\ [i^J\ Y_{Jm}]^*\ |I_i\ (M_i)\rangle \equiv (V.A.)\ \langle \gamma_5 Y_J\rangle \quad , \tag{A3}$$

$$\langle I_f\ (M_f)|[i^L\ \underset{\sim}{\sigma} \cdot \underset{\sim}{T}^{L}_{Jm}]^{\dagger}\ |I_i\ (M_i)\rangle \equiv (V.A.)\ \langle \underset{\sim}{\sigma} \cdot \underset{\sim}{T}^{L}_{J}\rangle \quad , \tag{A4}$$

and

$$\langle I_f\ (M_f)|[i^L\ \underset{\sim}{\alpha} \cdot \underset{\sim}{T}^{L}_{Jm}]^{\dagger}\ |I_i\ (M_i)\rangle \equiv (V.A.)\ \langle \underset{\sim}{\alpha} \cdot \underset{\sim}{T}^{L}_{J}\rangle \quad . \tag{A5}$$

Here complex-conjugate is denoted by a star and hermitian-conjugate by a dagger. The initial and final nuclear states are represented by $|I_i\ (M_i)\rangle$ and $|I_f\ (M_f)\rangle$; all nuclear variables other than spin quantum numbers are suppressed for simplicity in writing. With the conventions adopted in equations (A2)-(A5), all nuclear beta-moments, $\langle Y_J\rangle$, $\langle \gamma_5 Y_J\rangle$, $\langle \sigma \cdot T^L_J\rangle$, and $\langle \alpha \cdot T^L_J\rangle$, can be chosen to be simultaneously real (Konopinski 1963b).

The general parity-forbidden combination of nuclear matrix elements, $M_j(J)$, is given by the following equation:

$$M_j(J) \equiv C_V \left[\langle \underset{\sim}{\alpha} \cdot \underset{\sim}{T}_J^{J-1} \rangle + (\alpha Z/2j+1)\ (J/2J+1)^{\frac{1}{2}} \langle Y_J \rangle \right]$$

$$+ \ (\alpha Z/2j+1)\ (J+1/2J+1)^{\frac{1}{2}}\ C_A \langle \underset{\sim}{\sigma} \cdot \underset{\sim}{T}_J^J \rangle \quad . \qquad (A6)$$

The auxiliary quantities ρ_J and γ_o are defined by the following relations:

$$\rho_J(j,\bar{j}) = \left[\frac{(2j+1)\ (2\bar{j}+1)}{4\pi\ (2J+1)} \right]^{\frac{1}{2}} \langle j(-1/2)\ \bar{j}(+1/2)|J(0)\rangle \qquad (A7)$$

and

$$\gamma_o \equiv (1 - \alpha^2 Z^2)^{\frac{1}{2}} \qquad (A8)$$

The relativistic expression for the Fermi function is:

$$F(Z,W) = 2(1+\gamma_o)\ (2pR)^{2(\gamma_o-1)}\ e^{+\pi\eta} \left| \frac{\Gamma(\gamma_o + i\ \eta)}{\Gamma(2\gamma_o + i\ \eta)} \right|^2 , \qquad (A9)$$

where R is the nuclear radius, $\Gamma(w)$ is the Gamma-function of w, and $\eta = \alpha Z\ W/p$. When $(\alpha Z)^2 \ll 1$ or $W \gg 1$, then:

$$F(Z,W) \cong (2\pi\ \eta/1 - e^{-2\pi\eta}) \quad . \qquad (A10)$$

Equation (A9) differs by a factor of $(1 + \gamma_o)/2$ from the expression given in Bahcall (1962a).

APPENDIX B.

In this appendix, we give approximate analytic expressions for $K_1^{(2)}$, the generalized phase-space function that describes first-forbidden unique captures. We assume throughout this appendix that $(\alpha Z)^2 \ll 1$.

a) Nondegenerate Case

From equation (30), we see that for exoergic decays:

$$K_1^{(2)} \cong e^{+\nu} \int_0^{\infty} dp\; p^2 q^2\, F(Z,W)[p^2 + q^2]\; e^{-W/kT} \quad . \tag{B1}$$

Following Fowler and Hoyle (1963), we set

$$F(Z,W) = 2\pi\,\eta \left\langle \frac{F}{2\pi\,\eta} \right\rangle \tag{B2}$$

where $\eta = \alpha Z\, W/p$. The quantity $\langle F/2\pi\eta \rangle$ is a slowly varying function of temperature at high temperatures and hence can be estimated fairly accurately by evaluating F and η at some average energy, such as $(3\ kT/2)$.

Using (B2) in (B1), we find:

$$\begin{aligned} K_1^{(2)} = {} & 2\pi\,\alpha Z\, e^{+\nu-\beta} \left\langle \frac{F}{2\pi\,\eta} \right\rangle \beta^{-1} \Big[v^4 + 2v^2\,(v+1)^2\,\beta^{-1} \\ & + 2v(v+1)\,(v^2+7v+4)\,\beta^{-2} + 12(v+1)\,(2v^2+6v+1)\,\beta^{-3} \\ & + 24(7v^2+16v+6)\,\beta^{-4} + 720\,(v+1)\,\beta^{-5} + 1440\,\beta^{-6} \Big] \quad , \end{aligned} \tag{B3}$$

where

$$\beta = mc^2/kT \quad , \tag{B4}$$

$$v = \frac{W_o + mc^2}{mc^2} \quad , \tag{B5}$$

and

$$e^{-\nu} = \frac{1}{\pi^2 n_e \beta} \left(\frac{mc}{\hbar}\right)^3 K_2(\beta) \tag{B6}$$

Here $K_2(\beta)$ is a Bessel function of the second kind discussed on pages 79 and 202 of Watson's (1944) treatise and n_e is the electron concentration.

When $(mc^2/kT) \gg 1$, i.e., $T < 10^{+9}$ °K, then

$$K_1^{(2)} \cong \pi\alpha Z\, n_e \left(\frac{2\pi\,\hbar^2}{m\,k\,T}\right)^{3/2} \left\langle \frac{F}{2\pi\,\eta} \right\rangle \left(\frac{kT}{mc^2}\right) v^4 \quad , \tag{B7}$$

or, very approximately,

$$K_1^{(2)} \sim 0.1 \left\langle \frac{F}{2\pi\,\eta} \right\rangle \left(\frac{\pi\alpha Z}{\mu_e}\right) T_8^{-\frac{1}{2}}\, \rho_5\, v^4 \quad . \tag{B8}$$

Here T_8 is the temperature in units of 10^{+8} °K and ρ_5 is the density in units of 10^{+5} gm/cm^3.

The general result for exoergic decays ($P_o \neq 0$) is very complicated and we merely present two limiting expressions. If $|W_o| \gg kT$, then

$$K_1^{(2)} \cong 4\pi\,\alpha Z\,\beta^{-3} \left\langle \frac{F}{2\pi\,\eta} \right\rangle e^{+\nu - |W_o|/kT} (W_o P_o)^2 \tag{B9}$$

and if $|W_o| \ll kT$, then:

$$K_1^{(2)} \cong (2880\,\pi\alpha Z)\,\beta^{-7} \left\langle \frac{F}{2\pi\,\eta} \right\rangle e^{+\nu - |W_o|/kT} \quad . \tag{B10}$$

b) Degenerate Case

If $P_o = 0$ and the electrons are completely degenerate, then:

$$K_1^{(2)} = \frac{4\pi\,\alpha Z}{7} E_F^7 \left\langle \frac{F}{2\pi\,\eta} \right\rangle \left[1 + \frac{7(v+1)}{2E_F} + \frac{7}{10}\,\frac{(7v^2+16v+6)}{E_F{}^2} \right.$$

$$+ \frac{7(v+1)\,(2v^2+6v+1)}{4\,E_F{}^3} + \frac{7v(v+1)\,(v^2+7v+4)}{6\,E_F{}^4}$$

$$\left. + \frac{7v^2(v+1)}{2\,E_F{}^5} + \frac{7v^4}{2\,E_F{}^6} \right] , \qquad \text{(B11a)}$$

where the Fermi-kinetic-energy is given by:

$$E_F = 511 \left[\left\{ 1.02 \times 10^{-4} \left(\frac{\rho}{\mu_e} \right)^{2/3} + 1 \right\}^{\frac{1}{2}} - 1 \right] \text{kev}, \qquad \text{(B11b)}$$

and ρ is in gm/cm^3.

If $P_o \geq P_F$, then:

$$K_1^{(2)} = 0 \quad . \qquad \text{(B12)}$$

If $P_o \lesssim P_F$, i.e., $(W_F - |W_o|/W_F) \ll 1$, then:

$$K_1^{(2)} \cong 2\pi\,\alpha Z \left\langle \frac{F}{2\pi\,\eta} \right\rangle (W_o P_o)^2\,(W_F - |W_o|)^3/3 \quad . \qquad \text{(B13)}$$

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FIGURE CAPTION

Fig. 1. Astrophysically Important Levels of A^{40}, K^{40}, Ca^{40}.

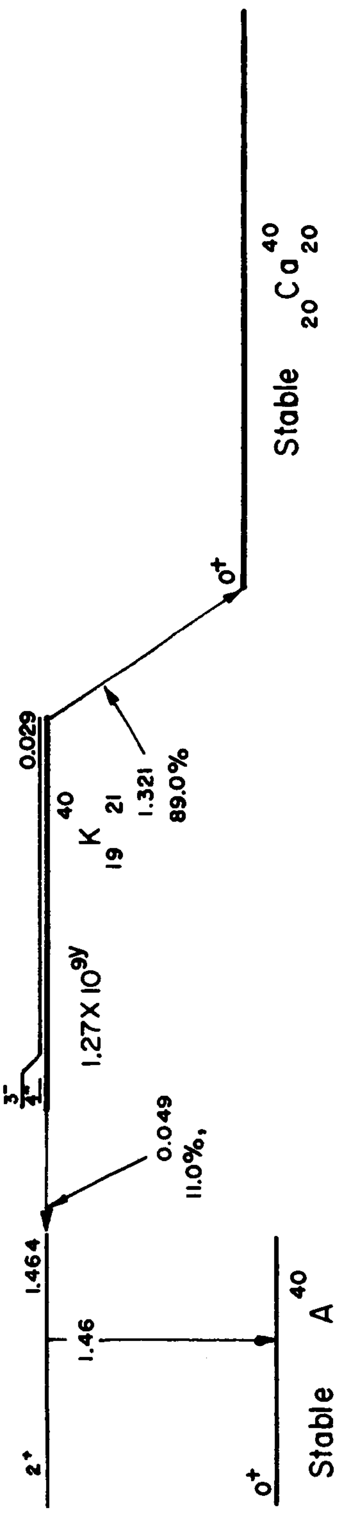
Stable $_{20}Ca^{40}_{20}$
0^+
0.029
$_{19}K^{40}_{21}$
1.321
89.0%
$1.27 X 10^{9y}$
3^-
4^-
0.049
11.0%,
1.464
1.46
2^+
0^+
Stable $_{18}A^{40}_{22}$

FIG. I